数理解析研究所講究録
第 1826 巻 2013 年 101-109



# $p$-adic Siegel Eisenstein series of degree $n$

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## 1 Introduction

In this paper, we define an Siegel Eisenstein series $G_{k,\chi}^{(n)}$ of degree $n$ and introduce a formula for its Fourier expansion. The definition of $G_{k,\chi}^{(n)}$ is different from the ordinary Siegel Eisenstein series $E_{k,\chi}^{(n)}$. But if $\chi$ satisfies a certain condition, $G_{k,\chi}^{(n)}$ coincides with $E_{k,\chi}^{(n)}$. We also introduce the theorem that states the existence of $\mathfrak{p}$-adic family of Siegel modular forms that interpolates $G_{k,\chi}^{(n)}$.

## 2 Statement of the main results

Let $F$ be a totally real field with $[F:\mathbf{Q}]=m$. If $K$ is a number field and $v$ is a finite place of $K$, then we denote by $\mathcal{O}_K$ and by $\mathcal{O}_v$ the integer ring of $K$ and that of $K_v$ respectively. For an ideal $\mathfrak{n}$ of $F$, we denote the group of fractional ideals of $F$ relatively prime to $\mathfrak{n}$ by $I_{\mathfrak{n}}$. Let $\chi$ be a narrow class character modulo $\mathfrak{n}$, that is, a character $\chi : I_{\mathfrak{n}} \to \mathbf{C}^{\times}$ trivial on any principal ideal $(a)$ generated by a totally positive element $a$ such that $a \equiv 1 \bmod \mathfrak{n}$. Let $\mathbb{A}_F$ be the adele ring of $F$ and $\mathbb{A}_F^{\times}$ the idele group of $F$. Denote the character of finite order of $\mathbb{A}_F^{\times}/F^{\times}$ corresponding to $\chi$ by $\widetilde{\chi}$.

For an infinite place $v$ of $F$, let $r_v$ be an element of $\mathbf{Z}/2\mathbf{Z}$ satisfying the following condition.

$$\chi((a)) = \prod_{v|\infty} \operatorname{sgn}(\iota_v(a))^{r_v} \text{ for } a \equiv 1 \bmod \mathfrak{n}.$$

Here $v$ runs over the set of $m$ real places of $F$ and $\iota_v$ is the real embedding corresponding to $v$. We define a character $\operatorname{sgn}_\chi$ of $F^{\times}$ by

$$\operatorname{sgn}_\chi(a) = \prod_{v|\infty} \operatorname{sgn}^{r_v}(\iota_v(a)).$$

We define a character $\chi_f : (\mathcal{O}_F/\mathfrak{n})^{\times} \to \mathbf{C}^{\times}$ by

$$\chi_f(a) = \operatorname{sgn}_\chi(a)\chi((a)).$$

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Let $n$ be a positive integer. For $0 \leq i \leq n$, we denote by $w_{n,i}$ the matrix given as follows.

$$w_{n,i} = \left(\begin{array}{cc|cc} 0_i & 0 & -1_i & 0 \\ 0 & 1_{n-i} & 0 & 0_{n-i} \\ \hline 1_i & 0 & 0_i & 0 \\ 0 & 0_{n-i} & 0 & 1_{n-i} \end{array}\right).$$

We put $w_n = w_{n,n}$. We define the symplectic group of degree $n$ by

$$\mathrm{Sp}_n(R) = \left\{ g \in \mathrm{GL}_{2g}(R) \mid {}^t g w_n g = w_n \right\},$$

where $R$ is a commutative ring. For $g \in \mathrm{Sp}_n$, we denote $g = \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix}$ with $a_g, b_g, c_g, d_g \in M_n$. Define the Siegel parabolic subgroup $P_n$ by

$$P_n(R) = \left\{ g \in \mathrm{Sp}_n(R) \mid c_g = 0 \right\}.$$

We define a congruence subgroup $\Gamma_0^{(n)}(\mathfrak{n})$ by

$$\Gamma_0^{(n)}(\mathfrak{n}) = \left\{ g \in \mathrm{Sp}_n(\mathcal{O}_F) \mid c_g \equiv 0 \bmod \mathfrak{n} \right\}.$$

We define the Siegel upper half space of degree $n$ by

$$\mathfrak{H}_n = \left\{ z \in \mathrm{Sym}_n(\mathbf{C}) \mid z = x + iy,\ x,\ y \in \mathrm{Sym}_n(\mathbf{R}),\ y > 0 \right\}.$$

Let $k$ be a positive integer and assume $\chi_f(-1) = (-1)^k$. Define a Siegel Eisenstein series of degree $n$, character $\chi$, weight $k$ by

$$E_{k,\chi}^{(n)}(z) = \sum_{g \in P_n(\mathcal{O}_F) \cap \Gamma_0^{(n)}(\mathfrak{n}) \backslash \Gamma_0^{(n)}(\mathfrak{n})} \chi_f^{-1}(\det d_g) \det(c_g z + d_g)^{-k}.$$

Here $z = (z_v)_{v|\infty} \in \prod_{v|\infty} \mathfrak{H}_n$ and $\det(c_g z + d_g)^{-k}$ is defined by

$$\det(c_g z + d_g)^{-k} = \prod_{v|\infty} \det(\iota_v(c_g) z_v + \iota_v(d_g))^{-k}.$$

In the rest of this paper, we assume that $\mathfrak{n}$ is relatively prime to 2 for simplicity.

Put

$$\mathcal{P} = \left\{ \mathfrak{p} : \text{a prime of } F \mid \mathfrak{p} \mid \mathfrak{n} \text{ and } \widetilde{\chi}_{\mathfrak{p}}^2 \text{ is unramified} \right\}.$$

Let $g \in \mathrm{Sp}_n(\mathcal{O}_F)$ and assume $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ with $0 \leq i_{\mathfrak{p}} \leq n$ if $\mathfrak{p} \in \mathcal{P}$. For $\mathfrak{p} \in \mathcal{P}$, the assumption for $g$ implies $g \bmod \mathfrak{p} \in P_n(\mathcal{O}_F/\mathfrak{p}) w_{i_{\mathfrak{p}}} P_n(\mathcal{O}_F/\mathfrak{p})$. Therefore if $\mathfrak{p} \in \mathcal{P}$, there exist elements $x_{\mathfrak{p}},\ y_{\mathfrak{p}} \in \mathrm{GL}(\mathcal{O}_F/\mathfrak{p})$ satisfying

$$g \bmod \mathfrak{p} = \begin{pmatrix} x_{\mathfrak{p}} & * \\ 0 & {}^t x_{\mathfrak{p}}^{-1} \end{pmatrix} w_{i_{\mathfrak{p}}} \begin{pmatrix} y_{\mathfrak{p}} & * \\ 0 & {}^t y_{\mathfrak{p}}^{-1} \end{pmatrix}.$$

We put

$$\chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;g) = \prod_{\substack{\mathfrak{p}|\mathfrak{n} \\ \mathfrak{p}\notin\mathcal{P}}} (\chi_f)_{\mathfrak{p}} (\det d_g) \prod_{\mathfrak{p}\in\mathcal{P}} (\chi_f)_{\mathfrak{p}} (\det x_{\mathfrak{p}} \det y_{\mathfrak{p}}) .$$

Here $(\chi_f)_{\mathfrak{p}}$ is the $\mathfrak{p}$-component of $\chi_f$.

We define an auxiliary Siegel Eisenstein series $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z)$ as follows.

$$E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z) = \sum_{g} \chi(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;g)^{-1} \det(c_g z + d_g),$$

where $g$ runs over the set $P_n(F) \cap \mathrm{Sp}_n(\mathcal{O}_F)\backslash\mathrm{Sp}_n(\mathcal{O}_F)$ satisfying the property $c_g \in \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}$ if $\mathfrak{p} \notin \mathcal{P}$ and $\mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}}(c_g \bmod \mathfrak{p}) = i_{\mathfrak{p}}$ if $\mathfrak{p} \in \mathcal{P}$. By the definition, we have $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z) = E^{(n)}_{k,\chi}$ if $i_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathcal{P}$.

Let $\mathfrak{p}$ be a prime of $F$ and assume $\mathfrak{p} \in \mathcal{P}$ and $(\mathfrak{p}, 2) = 1$. For $0 \le i \le n$ and $s \in \mathbf{C}$, we put $M_{in}(s, \widetilde{\chi}_{\mathfrak{p}}) = 0$ if $i$ is odd and put

$$M_{in}(s, \widetilde{\chi}_{\mathfrak{p}}) = \widetilde{\chi}_{\mathfrak{p}}(-1)\mathrm{N}\mathfrak{p}^{-i/2} \prod_{a=0}^{i/2} (1 - \mathrm{N}\mathfrak{p}^{-1-2a})(1 - \widetilde{\chi}^2_{\mathfrak{p}}(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-2s+2a+n-i-2}),$$

if $i$ is even. We set

$$m_i(k, \chi) = M_{in}(-k + \frac{n+1}{2}, \widetilde{\chi}_{\mathfrak{p}}).$$

By definition, the right hand side does not depend on $n$.

We define an Eisenstein $G^{(n)}_{k,\chi}$ as a linear combination of $E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z)$.

**Definition 2.1.** If $\mathcal{P} \neq \emptyset$, we define

$$G^{(n)}_{k,\chi}(z) = \sum_{\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}} \left( \prod_{\mathfrak{p}\in\mathcal{P}} m_{i_{\mathfrak{p}}}(k, \chi) \right) E'_{k,\chi}(\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}\,;z).$$

Here $\{i_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathcal{P}}$ runs over all the non-empty subsets of $\prod_{\mathfrak{p}\in\mathcal{P}} \{0, \ldots, n\}$. If $\mathcal{P} = \emptyset$, we define

$$G^{(n)}_{k,\chi} = E^{(n)}_{k,\chi}.$$

**Remark 2.1.** We can define $G^{(n)}_{k,\chi}$ more naturally by using the intertwining operator. But to shorten the statement, we define $G^{(n)}_{k,\chi}$ in this way. (See subsection 3.1)

The first main theorem of this paper is the result for Fourier coefficients for $G^{(n)}_{k,\chi}$. We prepare some notation.

Let $B \in \mathrm{Sym}^*_n(\mathcal{O}_F)$ be a half integral matrix of size $n$. Put $r = \mathrm{rank}\, B$. There exists a matrix $A \in \mathrm{GL}_n(F)$ such that

$${}^tABA = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix},$$

with $B' \in \mathrm{Sym}_r(F)$. Then $\det B' \in F^\times/F^{\times 2}$ does not depend on the choice of $A$. If $r$ is even we denote by $\chi_B$ the narrow class character of $F$ associated with the extension $F(\sqrt{(-1)^{r/2}\det B'})/F$ by the global class field theory.

For a prime $\mathfrak{p}$ of $F$ such that $\mathfrak{p} \nmid \mathfrak{n}$, there exists a matrix $U \in \mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ that satisfies

$$ {}^t U B U = \begin{pmatrix} B'_\mathfrak{p} & 0 \\ 0 & 0 \end{pmatrix}, $$

with $B'_\mathfrak{p} \in \mathrm{Sym}^*_r(\mathcal{O}_\mathfrak{p})$. The matrix $B'_\mathfrak{p}$ is unique up to unimodular equivalence. Therefore $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p};T)$ does not depend on the choice of $U$, where $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p};T)$ is the polynomial obtained by the Siegel series. (In the notation of [4] 13.6. Theorem, we have $\Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p};T) = f_{B'_\mathfrak{p}}(T)$.) Thus we put $\Phi^{(r)}_\mathfrak{p}(B,T) = \Phi^{(r)}_\mathfrak{p}(B'_\mathfrak{p},T)$.

**Theorem 2.1.** *Let $0 \le B \in \mathrm{Sym}^*_n(\mathcal{O}_F)$ be a half integral positive semi-definite matrix of size $n$ and $k > n+1$ an integer. Let $\chi$ be a primitive narrow class character of $F$ of conductor $\mathfrak{n}$. Put $r = \operatorname{rank} B$. Then the following assertions hold.*

*If $r$ is even, then $a(B, G^{(n)}_{k,\chi})$ is given by*

$$ 2^{rm/2} \prod_{\mathfrak{p}\nmid\mathfrak{n}} \Phi^{(r)}_\mathfrak{p}(B;\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1}) $$

$$ \times L(1-k,\chi)^{-1} L^{(\mathfrak{n})}(1+r/2-k,\chi_h\chi) \prod_{i=1}^{r/2} L^{(\mathfrak{n})}(1+2i-2k,\chi^2)^{-1}. $$

*If $r$ is odd, then $a(B, G^{(n)}_{k,\chi})$ is given by*

$$ 2^{(r+1)m/2} \prod_{\mathfrak{p}\nmid\mathfrak{n}} \Phi^{(r)}_\mathfrak{p}(B;\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{k-r-1}) $$

$$ \times L(1-k,\chi)^{-1} \prod_{i=1}^{(r-1)/2} L^{(\mathfrak{n})}(1+2i-k,\chi^2)^{-1}. $$

*For a Hecke L-function $L(s,\chi)$ and an ideal $\mathfrak{n}$, we denote $L^{(\mathfrak{n})}(s,\chi) = \prod_{\mathfrak{p}\nmid\mathfrak{n}}(1 - \chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s})^{-1}$, where the index $\mathfrak{p}$ runs over the set of primes of $F$ relatively prime to the conductor of $\chi$ and the ideal $\mathfrak{n}$.*

From this theorem and the definition of $G^{(n)}_{k,\chi}$, we have a formula for the Fourier coefficients of $E^{(n)}_{k,\chi}$ if $\mathcal{P} = \emptyset$.

**Remark 2.2.** When $F = \mathbf{Q}$, Katsurada [1] proved the explicit formula for $\Phi^{(r)}_\mathfrak{p}(B,T)$, thus in this case we have the explicit formula for Fourier coefficients of $G^{(n)}_{k,\chi}$.

For a prime $\mathfrak{p}$ of $F$, we put

$$ P_+(\mathfrak{p}^\alpha) = \left\{ a\mathcal{O}_F \mid a\text{:positive definite and } a \equiv \mathrm{mod}\,\mathfrak{p}^\alpha \right\}. $$

We define the narrow ray class group of $F$ of conductor $\mathfrak{p}^\alpha$ by $\mathrm{Cl}_F(\mathfrak{p}^\alpha) = I_{\mathfrak{p}}/P_+(\mathfrak{p}^\alpha)$. We consider the projective limit

$$\mathrm{Cl}_F(\mathfrak{p}^\infty) = \varprojlim \mathrm{Cl}_F(\mathfrak{p}^\alpha).$$

We put $G = \mathrm{Cl}_F(\mathfrak{p}^\infty)$.

Let $\Omega$ be the completion of $\overline{F}_{\mathfrak{p}}$ and $A$ the integer ring of $\Omega$. We fix the embedding of $\overline{\mathbf{Q}}$ to $\mathbf{C}$ and $\Omega$. We denote $\mathrm{Meas}(G, A)$ by the bounded $\mathfrak{p}$-adic measure on $G$ with values in $A$. Let $p$ be the residual characteristic of $F_{\mathfrak{p}}$. Since $I_{\mathfrak{p}}$ can be considered as a dense subgroup of $G$ and the norm map $N : I_{\mathfrak{p}} \to \mathbf{Z}_p^\times$ is continuous, we can extend $N$ to $G$. We denote the extended character by the same letter. Let $\omega$ be the Teichmüller character of $\mathbf{Z}_p^\times$ and put $\omega_F = \omega \circ N$.

**Theorem 2.2.** *Let $\mathfrak{p}$ be a prime of $F$ such that $(\mathfrak{p}, 2) = 1$, $p$ a residual characteristic of $F_{\mathfrak{p}}$ and $\chi$ a narrow ray class character of conductor $\mathfrak{p}^\nu$. Denote $\mathcal{O}_F[\chi]$ by the ring generated by $\mathrm{Im}(\chi)$ over $\mathcal{O}_F$. Then there exists a formal Fourier expansion $\mathbf{G}^{(n)}(\chi; T)$*

$$\mathbf{G}^{(n)}(\chi; T) = \sum_{0 \leq B \in \mathrm{Sym}_n^{(*)}(\mathcal{O}_F)} \mathbf{a}(B; T)\mathbf{e}(Bz),$$

*where $\mathbf{a}(B; T)$ is an element of the quotient ring of the formal power series ring $\mathrm{Frac}\,\mathcal{O}_F[\chi][\![T]\!]$, and satisfies the following condition. If $k > n+1$ and $\chi \cdot \omega_F^{-k}$ is not the trivial character modulo $\mathfrak{p}$*

$$\mathbf{G}^{(n)}(\chi; u^k - 1) = G_{k, \chi \cdot \omega_F^{-k}},$$

*where $u$ is a fixed generator of $1 + \mathbf{Z}_p$. Moreover, there exists a nonzero formal power series $\mathbf{b}(T)$ and a $\mathfrak{p}$-adic measure $\mu_B \in \mathrm{Meas}(G, A)$ for each $B$ that satisfy*

$$\mathbf{b}(u^s - 1)\mathbf{a}(B; u^s - 1) = \int_G \chi(x) N(x)^{-1} \langle N(x) \rangle^s d\mu_B, \quad \textit{for } s \in \mathbf{Z}_p.$$

*Here for $a \in \mathbf{Z}_p^\times$, we put $\langle a \rangle = a\omega^{-1}(a)$.*

**Remark 2.3.** In the interpolation property, we assume the character is not trivial character mod $\mathfrak{p}$. H. Kawamura [2] proved the existence $p$-adic family of Siegel Eisenstein series that interpolates Eisenstein series with trivial character modulo $p$.

# 3 Sketch of the proof of the main theorem

Since we can derive theorem 2.2 by theorem 2.1 and the existence of $\mathfrak{p}$-adic Heck $L$-functions for totally real fields, we only prove theorem 2.1.

### 3.1 Definition of an Eisenstein series $\widetilde{G}^{(n)}_{k,\chi}$

In this subsection, we give a more natural definition of $G^{(n)}_{k,\chi}$.

For a place $v$ of $F$, we denote the space for the normalized induction by

$$\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^{s}).$$

We define the intertwining operator

$$M^{(s)}_{w_n} : \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v|\cdot|_v^{s}) \to \mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v^{-1}|\cdot|_v^{-s})$$

by

$$M^{(s)}_{w_n}(f)(g) = \int_{\mathrm{Sym}_n(F_v)} f\left(w_n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx,$$

for $g \in \mathrm{Sp}_n(F_v)$. Here we take a Haar measure of $\mathrm{Sym}_n(F_v)$ so that we have $\int_{\mathrm{Sym}_n(\mathcal{O}_v)} dx = 1$. The integral is convergent if $\mathrm{Re}\, s$ is sufficiently large and has meromorphic continuation to the whole complex plane.

We define a compact subgroup $C_{0,v}$ of $\mathrm{Sp}_n(F_v)$ as follows.

(i) If $v$ is real or $\widetilde{\chi}_v$ is unramified then we define $C_{0,v} = C_v$.

(ii) If $v = \mathfrak{p}$ is a finite place and $\widetilde{\chi}_{\mathfrak{p}}$ is ramified then we define

$$C_{0,v} = \left\{ g \in \mathrm{Sp}_n(\mathcal{O}_v) \mid c_g \equiv 0 \bmod \mathfrak{p}^{\nu} \right\}.$$

Here $\mathfrak{p}^{\nu}$ is the conductor of $\widetilde{\chi}_v$.

We define a character $\kappa_v$ of $C_{0,v}$ as follows.

(i) If $v$ is real then we define

$$\kappa_v\left(\begin{pmatrix} u & -v \\ v & u \end{pmatrix}\right) = \det(u + iv)^{-k}.$$

(ii) If $v$ is finite and $\widetilde{\chi}_v$ is unramified then we define $\kappa_v = 1$.

(iii) If $v$ is finite and $\widetilde{\chi}_v$ is ramified then we define

$$\kappa_v(\gamma) = \widetilde{\chi}_v(\det d_\gamma).$$

We denote by $\phi_v(s, \cdot)$ the element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v|\cdot|_v^{s})$ satisfying the following conditions.

$$\begin{aligned} &\mathrm{supp}\phi_v(s,\cdot) = P_n(F_v)C_{0,v}, \\ &\phi_v(s, g\gamma) = \kappa_v(\gamma)\phi_v(s,g) \quad \text{for all } \gamma \in C_{0,v}, \\ &\phi_v(s,1) = 1. \end{aligned}$$

We also denote by $\phi'_v(-s,\cdot)$ the element of $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v^{-1}|\cdot|_v^{-s})$ satisfying the following conditions.

$$\begin{aligned}&\mathrm{supp}\phi'_v(-s,\cdot)=P_n(F_v)w_nC_{0,v},\\&\phi'_v(-s,g\gamma)=\kappa_v(\gamma)\phi'_v(-s,g)\quad\text{for all }\gamma\in C_{0,v}.\\&\phi'_v(-s,w_n)=1.\end{aligned}\tag{3.1}$$

For a place $v$ of $F$, we define $\varphi_v(s,\cdot)\in\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_v|\cdot|_v^s)$ as follows.

(i) If $v$ is real or $\widetilde{\chi}_v$ is unramified then we define

$$\varphi_v(s,g)=\phi_v(s,g).$$

(ii) If $v$ is finite and $\widetilde{\chi}_v$ is ramified then we define

$$\varphi_v(s,g)=M_{w_n}^{(-s)}(\phi'_v(-s,\cdot))(g).\tag{3.2}$$

For $g=(g_v)_v\in\mathrm{Sp}_n(\mathbb{A}_F)$, we put

$$\varphi(s,g)=\prod_v\varphi_v(s,g_v),$$

where $v$ runs over the set of the places of $F$.

We define an Eisenstein series on $\mathrm{Sp}_n(\mathbb{A}_F)$ by

$$\widetilde{\mathcal{G}}_{s,\chi}^{(n)}(g)=\sum_{\gamma\in P_n(F)\backslash\mathrm{Sp}_n(F)}\varphi(s,\gamma g).$$

We define $\widetilde{G}_{k,\chi}^{(n)}$ by the function on $\prod_{v|\infty}\mathfrak{H}_n$ corresponding to $\widetilde{\mathcal{G}}_{k-(n+1)/2,\chi}^{(n)}$.

We can prove the proposition bellow by explicit calculation of the value of the intertwining operator. We omit the proof.

**Proposition 3.1.** *Assume that* $\mathfrak{n}$ *is relatively prime to* 2. *Then we have*

$$G_{k,\chi}^{(n)}=\widetilde{G}_{k,\chi}^{(n)}.$$

## 3.2 Functional equation of Whittaker functions

The key ingredient for the proof of the main theorem is the following theorem by T. Ikeda (Kyoto University).

**Theorem 3.1** (T. Ikeda). *Let $k$ be a local field. Let $\psi$ be a nontrivial additive character of $k$ and $\chi$ be a quasi character of $k^\times$. Suppose $B\in\mathrm{Sym}_n(k)$ and* $\det B\neq 0$. *For $f\in\mathrm{Ind}_{P_n}^{\mathrm{Sp}_n}(\widetilde{\chi}_{\mathfrak{p}}|\cdot|_{\mathfrak{p}}^s)$, we put*

$$W_B(f)(g)=\int_{\mathrm{Sym}_n(k)}f\left(w_n\begin{pmatrix}1_n&x\\0_n&1_n\end{pmatrix}\right)\psi(-\mathrm{Tr}Bx)dx.$$

*Then*

$$W_B \circ M_{w_n} = \chi(\det B)^{-1}|\det B|^{-s}c(s,B)W_B.$$

*The notation is as follows.*

*Let* $n$ *be even.* $D_B$ *is defined by* $D_B = (-1)^{n/2}\det B$. $\chi_B$ *is the character of* $k^\times$ *corresponding to* $k(\sqrt{D_B})/k$. $c(s,B)$ *is given as follows.*

$$c(s,B) = |2|^{-ns}\frac{\alpha(D_B)}{\alpha(1)}\chi(2)^{-n}\varepsilon'(s+\frac{1}{2},\chi\chi_B,\psi)$$
$$\times\,\varepsilon'(s-\frac{n-1}{2},\chi,\psi)^{-1}\prod_{r=1}^{n/2}\varepsilon'(2s-n+2r,\chi^2,\psi)^{-1}.$$

*Here* $= \varepsilon(s,\omega,\psi)$ *is the epsilon factor,* $\varepsilon'(s,\omega,\psi) = \varepsilon(s,\omega,\psi)\frac{L(1-s,\omega^{-1})}{L(s,\omega)}$ *and* $\alpha(*)$ *is the Weil index.*

$$\frac{\alpha(1)}{\alpha(D_B)} = \varepsilon(\frac{1}{2},\chi_B,\psi).$$

*Let* $n$ *be odd. Then*

$$c(s,B) = |2|^{-(n-1)s}\chi(2)^{-(n-1)}\zeta_B$$
$$\times\,\varepsilon'(s-\frac{n-1}{2},\chi,\psi)^{-1}\prod_{r=1}^{(n-1)/2}\varepsilon'(2s-n+2r,\chi^2,\psi)^{-1}.$$

*Here*

$$\zeta_B = ((-1)^{(n-1)/2},\det B)(-1,-1)^{(n^2-1)/8}h(B),$$

*and* $(*,*)$ *is the Hilbert symbol.*

## 3.3 Sketch of the proof

*sketch of the proof of theorem 2.1.* For simplicity, we assume the class number of $F$ is one. Denote $\Phi$ by the Siegel operator. Then by the definition of $G^{(n)}_{k,\chi}$, we have $\Phi G^{(n)}_{k,\chi} = G^{(n-1)}_{k,\chi}(z)$. Thus it is enough to compute $a(B,G^{(n)}_{k,\chi})$ when $\det B \neq 0$. We can prove that $a(B,G^{(n)}_{k,\chi})$ has Euler product expression. By [3] (4.34K), (4.35K), [4] 13.6. Theorem, we know the Euler factor at infinite places and unramified places. Thus it is enough to compute the Euler factors at ramified places. Let $\mathfrak{p} \mid \mathfrak{n}$. Then the Euler factor at $\mathfrak{p}$ is given by

$$\int_{\mathrm{Sym}_n(F_\mathfrak{p})}\varphi_v\left(k-(n+1)/2, w_n\begin{pmatrix}1_n & x\\ 0_n & 1_n\end{pmatrix}\right)\mathrm{e}(-\mathrm{Tr}Bx)dx$$
$$= W_B \circ M_{w_n}(\phi'(-k+(n+1)/2,\cdot))(1_n).$$

By theorem 3.1, it is enough to compute $W_B(\phi'(-k+(n+1)/2,\cdot))(1_n)$ , but it is easy to verify that this equals to 1. □

## References

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